Molecular Crystals and Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl16

Equivalent Permittivity Tensor for Parallel Anisotropic Homogeneous Stratified Media

David S. Moroi [a]

[a] Department of Physics, Kent State University, Kent, Ohio, 44242

Version of record first published: 28 Mar 2007.

To cite this article: David S. Moroi (1972): Equivalent Permittivity Tensor for Parallel Anisotropic Homogeneous Stratified Media, Molecular Crystals and Liquid Crystals, 18:3-4, 327-332

To link to this article: http://dx.doi.org/10.1080/15421407208083604

Molecular Crystals and Liquid Crystals. 1972. Vol. 18, pp. 327–332

Printed in Great Britain

Equivalent Permittivity Tensor for Parallel Anisotropic Homogeneous Stratified Media†‡

DAVID S. MOROI

Department of Physics
Kent State University
Kent, Ohio 44242

Received February 8, 1972

Abstract—The equivalent permittivity tensor for a medium which consists of any number of parallel layers of arbitrary anisotropic homogeneous materials is calculated in the long wavelength limit. The formal results of the calculation are shown to be identical with the previous results obtained for the special case in which the system has a periodic structure. If the principal axes of the permittivity tensor in each successive layer have the same orientation, the expressions for the principal values of the equivalent permittivity tensor are similar to those for a number of capacitors in series or in parallel.

1. Introduction

To investigate the form birefringence of some smectic liquid crystals, we recently calculated the effective dielectric tensor (in the long wavelength limit) of a stratified medium.[1] This medium was defined to be a periodic arrangement of parallel layers of thin homogeneous anisotropic materials. The purpose of this paper is to generalize the work of Ref. 1; i.e. to calculate, in the long wavelength limit, the equivalent permittivity tensor for a stratified medium of finite thickness consisting of any number of parallel layers of arbitrary anisotropic homogeneous materials which do not have periodic arrangement of layers.[2]

2. Calculation

In the present calculation, we are interested only in the electric properties of a medium which consists of any number (l) of parallel

† Research supported in part by a 1971 Kent State University Summer Research Fellowship.

‡ Work performed at the Aspen Center for Physics.

layers of arbitrary anisotropic homogeneous materials of permittivity tensor $\epsilon^{(i)}$ and thickness t_i ($i = 1, 2, \ldots, l$) as shown in Fig. 1. In order to simplify the calculation, we assume that the magnetic permeability tensors, $\mu^{(i)}$, are set equal to unity and the field in a layer does not vary appreciably over the layer (i.e. the long wavelength approximation).[1]

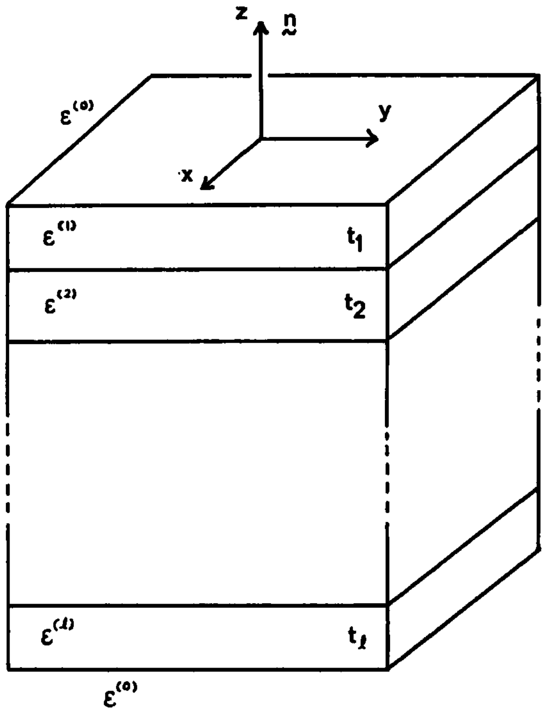

Figure 1. A stratified medium which consists of l parallel layers of arbitrary anisotropic homogeneous materials of permittivity tensor $\epsilon^{(i)}$ and thickness t_i ($i = 1,2, \ldots, l$). **n** is the unit vector normal to the interface.

Let us denote the electric field and displacement vectors in the i-th layer by $\mathbf{E}_i$ and $\mathbf{D}_i$. In the long wavelength approximation, then, the average electric field $\mathbf{E}$ and the average electric displacement $\mathbf{D}$ over the entire medium is given by

$$\mathbf{E} = \sum_{i=1}^{l} f_i \mathbf{E}_i \tag{1}$$

and

$$\mathbf{D} = \sum_{i=1}^{l} f_i \mathbf{D}_i, \tag{2}$$

where

$$\mathbf{D}_i = \epsilon^{(i)} \mathbf{E}_i, \tag{3}$$

and

$$f_i = t_i / (\sum_{j=1}^{l} t_j). \tag{4}$$

The boundary conditions[3] which the fields must satisfy are

$$\left.\begin{aligned} \mathbf{n} \times (\mathbf{E}_{i-1} - \mathbf{E}_i) &= 0, \\ \text{and} \qquad \mathbf{n} \cdot (\mathbf{D}_{i-1} - \mathbf{D}_i) &= 0. \end{aligned}\right\} \tag{5}$$

From Eqs. (3) and (5), it follows that

$$\mathbf{E}_i = a_i \mathbf{E}_{i-1} \tag{6}$$

where

$$a_i = 1 + e_i^{-1} \mathbf{nn} \{\epsilon^{(i-1)} - \epsilon^{(i)}\} \tag{7}$$

and

$$e_i = \mathbf{n} \cdot \epsilon^{(i)} \cdot \mathbf{n}.$$

Here the expression **nn** denotes the dyadic of the unit normal vector **n**. Equations. (6) are simple recurrence relations which indicate that $\mathbf{E}_i$ can be expressed in terms of $\mathbf{E}_0$ which is the electric field outside the medium; i.e.

$$\mathbf{E}_i = A_i \mathbf{E}_0, \tag{8}$$

where

$$A_i = a_i a_{i-1} \ldots a_2 a_1. \tag{9}$$

Some straightforward algebra gives

$$A_i = 1 + e_i^{-1} \mathbf{nn} (\epsilon^{(0)} - \epsilon^{(i)}). \tag{10}$$

Substituting Eq. (8) into Eq. (1) one has

$$\mathbf{E} = (\sum_{i=1}^{l} f_i A_i) \mathbf{E}_0. \tag{11}$$

With the help of Eqs. (3) and (8), Eq. (2) can be written

$$\mathbf{D} = (\sum_{i=1}^{l} f_i \epsilon^{(i)} A_i) \mathbf{E}_0 \tag{12}$$

Let us now define the equivalent permittivity tensor ϵ of the medium by the equation

$$\mathbf{D} = \epsilon \mathbf{E}. \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (13) and noting that the external electric field $\mathbf{E}_0$ can be chosen arbitrarily, we obtain

$$\epsilon = \{\sum_{i=1}^{l} f_i \epsilon^{(i)} A_i\}\{\sum_{j=1}^{l} f_j A_j\}^{-1}. \tag{14}$$

Some straightforward but tedious algebra gives rise to:

$$\epsilon = \sum_{i=1}^{l} f_i \epsilon^{(i)}\{1 - e_i^{-1}\mathbf{nn}\epsilon^{(i)}\} + (\sum_{i=1}^{l} f_i/e_i)^{-1} \sum_{j,k=1}^{l} (f_j f_k/e_j e_k)\epsilon^{(j)}\mathbf{nn}\epsilon^{(k)}. \tag{15}$$

A further manipulation yields:

$$\epsilon = \sum_{i=1}^{l} f_i \epsilon^{(i)} - (\sum_{i=1}^{l} f_i/e_i)^{-1} \sum_{j<k} (f_j f_k/e_j e_k)(\epsilon^{(j)} - \epsilon^{(k)})\mathbf{nn}(\epsilon^{(j)} - \epsilon^{(k)}). \tag{16}$$

Equation (16) is identical to the expression for the effective permittivity tensor for a periodic stratified medium.[1]

3. Discussion

It is interesting to note that the expresion for ϵ in Eq. (16) has exactly the same form as the effective permittivity tensor for a periodic stratified medium that consists of an unlimited number of l alternate plane layers of arbitrary anisotropic homogeneous materials.[1] The reason why the two expressions are the same is that the medium with periodicity is considered to be made up of an unlimited number of identical layers of the equivalent permittivity tensor ϵ, each of which consists of a given number of parallel layers of arbitrary anisotropic homogeneous materials. In fact, one can easily show that the permittivity tensor for an infinite stratified medium with a periodic structure can be obtained from Eq. (15). That is because the medium has a periodic arrangement, we set

$$l = mn, \tag{17}$$

where m is an integer which eventually may be allowed to approach infinity and n is the number of layers consisting of different materials.

However, we still retain the long wavelength approximation; i.e. the thickness of each of the n-layers is much less than the wavelength of light. The f_i's in Eq. (15) now become

$$f_i = t_i/(m \sum_{j=1}^{n} t_j) = g_i/m, \tag{18}$$

where

$$g_i = t_i/(\sum_{j=1}^{n} t_j). \tag{19}$$

Substituting Eqs. (17) and (18) into Eq. (15) and noting the cancelation of m because of

$$\sum_{i=1}^{l} \cdots = m \sum_{i=1}^{n} \cdots \text{ and } \sum_{j,k=1}^{l} \cdots = m^2 \sum_{j,k=1}^{n} \cdots,$$

we have

$$\epsilon = \sum_{i=1}^{n} g_i \epsilon^{(i)}\{1 - e_i^{-1}\mathbf{nn}\epsilon^{(i)}\} + (\sum_{i=1}^{n} g_i/e_i)^{-1} \sum_{j,k=1}^{n} (g_j g_k/e_j e_k)\epsilon^{(j)}\mathbf{nn}\epsilon^{(k)}. \tag{20}$$

The ϵ given in Eq. (20) can be reduced to the same form as the expression given in Eq. (16).

As pointed out in the previous paper of Ref. 1, if the principal axes of the tensor $\epsilon^{(i)}$ in each layer have the same orientation and, further, one of the principal axes is chosen to be parallel to the unit vector $\mathbf{n}$ (the z-axis) perpendicular to the interface, then the equivalent permittivity tensor ϵ takes a very simple form

$$\epsilon = \begin{bmatrix} \sum_{i=1}^{l} f_i \epsilon_x^{(i)} & 0 & 0 \\ 0 & \sum_{i=1}^{l} f_i \epsilon_y^{(i)} & 0 \\ 0 & 0 & (\sum_{i=1}^{l} f_i/\epsilon_z^{(i)})^{-1} \end{bmatrix}. \tag{21}$$

From the pedagogical point of view, it is very interesting to observe the expression for ϵ given by Eq. (21) by the following analogy: The expression for the principal values of ϵ in the x- and y-directions which are parallel to the interface are very similar to those for a given number of capacitors connected in parallel. The expression for the principal value of ϵ in the z-direction, which is perpendicular to the interface, is also very similar to that for a given number of capacitors connected in series.

Acknowledgements

The author wishes to thank the Kent State University for its financial support and the Aspen Center for Physics for its hospitality during his stay there. He also thanks Dr. David Uhrich and Dr. Wilbur Franklin for a critical reading of the manuscript and many helpful discussions.

REFERENCES

1. Cvikl, B., Moroi, D. and Franklin, W., *Mol. Cryst. and Liq. Cryst.* **12**, 267 (1971).
2. Moroi, D. S.; a part of this work was reported at the 1971 Fall Meeting of the Ohio Section of the American Physical Society which was held at the University of Toledo, Toledo, Ohio 43606, on October 15, 1971.
3. Born, M. and Wolf, E., *Principles of Optics*, 4th edition, Pergamon Press, 1970, p. 5.